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LETTER TO THE EDITOR

Universal amplitudes in finite-size scaling: generalisation to arbitrary dimensionality

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Abstract. The relationship between the correlation length and critical exponents in finite width strips in two dimensions is generalised to cylindrical geometries of arbitrary dimensionality d. For $d \neq 2$ these correspond, however, to curved spaces. The result is verified for the spherical model.

A striking result of conformal invariance at critical points has been the relationship between the amplitude A of the finite-size scaling behaviour of the correlation length ξ of an infinitely long strip of width L with periodic boundary conditions, defined by $\xi^{-1} \sim A/L$, and the scaling dimension x of the corresponding scaling operator (Cardy 1984). This relation states that

$$A = 2\pi x. \tag{1}$$

It has been verified in a large number of two-dimensional models (Luck 1982, Derrida and de Seze 1982, Nightingale and Blote 1983, Privman and Fisher 1984, Penson and Kolb 1984, Alcaraz *et al* 1985), and gives a very accurate way of determining the scaling dimensions numerically. It would therefore be very useful to generalise this result to dimensionality $d \neq 2$. In this letter we describe such a generalisation. Unfortunately the result appears to be difficult to utilise for numerical work.

For the purposes of generalisation, it is convenient first to restate the two-dimensional conformal invariance argument in a fashion independent of the use of complex variables. Consider a critical theory defined on the infinite two-dimensional plane $\mathbb{R}^2$. In polar coordinates, the metric is

$$ds^2 = dr^2 + r^2\, d\theta^2. \tag{2}$$

Under the coordinate transformation

$$(r, \theta) = (\exp(u/R), \theta) \tag{3}$$

where $-\infty < u < \infty$, the metric can be written

$$ds^2 = R^{-2} \exp(2u/R)(du^2 + R^2\, d\theta^2) \tag{4}$$

which can be recognised as a conformal factor multiplying the natural metric for the space $S^1 \times \mathbb{R}^1$, i.e. the surface of a circular cylinder of radius R. The transformation (3) thus conformally relates theories defined on $\mathbb{R}^2$ and $S^1 \times \mathbb{R}^1$. At a critical point, a

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two-point correlation function transforms according to

$$\langle\varphi(\exp(u_1/R),\theta_1)\varphi(\exp(u_2/R),\theta_2)\rangle_{\mathbb{R}^2}$$
$$=R^{2x}\exp[-x(u_1+u_2)/R]\langle\varphi(u_1,\theta_1)\varphi(u_2,\theta_2)\rangle_{S^1\times\mathbb{R}^1}. \tag{5}$$

Since the correlation function on the left-hand side is proportional to $|\boldsymbol{r}_1-\boldsymbol{r}_2|^{-2x}$, this determines the correlation function on the cylinder and gives a correlation length $\xi=R/x$. This agrees with (1) if we note that $L=2\pi R$.

The generalisation to $d\neq 2$ is now straightforward. The metric on $\mathbb{R}^d$ is written

$$\mathrm{d}s^2=\mathrm{d}r^2+r^2\,\mathrm{d}\Omega^2 \tag{6}$$

where, for example, in $d=3$,

$$\mathrm{d}\Omega^2=\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\psi^2. \tag{7}$$

Under (3) the metric is transformed conformally into

$$\mathrm{d}s^2=R^{-2}\exp(2u/R)(\mathrm{d}u^2+R^2\,\mathrm{d}\Omega^2) \tag{8}$$

which, apart from a conformal factor, is the natural metric for $S^{d-1}\times\mathbb{R}^1$. Thus critical theories on $\mathbb{R}^d$ are conformally related to those on this cylindrical geometry. As before, the correlation length along the cylinder is given by $\xi=R/x$.

This result is supposed to be valid in the continuum limit, at the critical point. For it to be useful for numerically estimating the scaling dimension x, it is necessary to approximate the continuum by a sequence of lattices. It is most convenient to choose these lattices to be regular. For $d\neq 2$, however, the space $S^{d-1}\times\mathbb{R}^1$ is curved, and only a finite number of regular lattices may be embedded in the space. For S^2, for example, these correspond to the Platonic solids, the largest of which is the dodecahedron (12 lattice points). It is not clear whether this approximates the continuum sufficiently well to give accurate values for the exponents. In addition, the lattice approximation to $S^{d-1}\times\mathbb{R}^1$ should incorporate those symmetries which correspond to translations in $\mathbb{R}^d$, and mix up the spaces S^{d-1} and $\mathbb{R}^1$. This will be true if, on distance scales much less than the inverse curvature R, the lattice is isotropic in all d directions. If this last requirement is relaxed, by introducing a lattice which approximates the symmetries of S^{d-1} only, then the amplitudes A will not be universal, but ratios of them will be. In this case it may be simplest to take the anisotropic limit, and to consider an equivalent quantum Hamiltonian defined on a lattice approximating S^{d-1}, as has already been done for $d=2$ (Penson and Kolb 1984, Alcaraz *et al* 1985).

It would of course be useful to obtain a formula for A for geometries which are more easily approximated by a lattice, for example a 2-torus $\times\mathbb{R}^1$. Such a formula cannot be obtained by conformal transformation, because these spaces are flat, and for $d\neq 2$ the group of conformal transformations of flat space into itself is too restricted.

The only simple test of our result we have been able to find is in the spherical model, equivalent to the $n\to\infty$ limit of the n-vector model, defined by the reduced Hamiltonian

$$\mathscr{H}=\int[\tfrac{1}{2}(\nabla\boldsymbol{\varphi})^2+\tfrac{1}{2}\mu_0^2\boldsymbol{\varphi}^2+\tfrac{1}{4}(\lambda_0 n^{-1})(\boldsymbol{\varphi}^2)^2]\,\mathrm{d}^dx. \tag{9}$$

For $d=3$ the unrenormalised propagator on $S^2\times\mathbb{R}^1$ is found by decomposing $\boldsymbol{\varphi}(u,\theta,\psi)$ into normal modes $\mathrm{e}^{\mathrm{i}ku}Y_{lm}(\theta,\psi)$. The result is

$$G^0_{lm}(k)=[k^2+R^{-2}l(l+1)+\mu_0^2]^{-1}. \tag{10}$$

In the $n \to \infty$ limit the only effect of the interactions (Ma 1976) is to replace μ_0^2 by μ^2, where μ^2 is determined self-consistently by

$$\mu^2 = \mu_0^2 + \frac{\lambda_0}{2} \int \frac{\mathrm{d}k}{2\pi} \frac{1}{4\pi R^2} \sum_{lm} \frac{1}{k^2 + R^{-2} l(l+1) + \mu^2}. \tag{11}$$

The sum over m gives a factor of $2l+1$. The values of μ_0^2 corresponding to the bulk critical point is given by

$$0 = \mu_0^2 + \frac{\lambda_0}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{1}{k^2 + k_\perp^2} \tag{12}$$

so that the correlation length in the cylindrical geometry is $\xi = \mu^{-1}$ where

$$\mu^2 = \frac{\lambda_0}{2} \int \frac{\mathrm{d}k}{2\pi} \left(\frac{1}{4\pi R^2} \sum_l \frac{2l+1}{k^2 + R^{-2} l(l+1) + \mu^2} - \int \frac{\mathrm{d}^2 k_\perp}{(2\pi)^2} \frac{1}{k^2 + k_\perp^2} \right). \tag{13}$$

The sums over modes are separately well defined only with an ultraviolet cut-off

$$k^2 + k_\perp^2 < \Lambda^2 \qquad k^2 + R^{-2} l(l+1) < \Lambda^2. \tag{14}$$

Equation (13) determines μ^2 as a function of R and Λ. For $R\Lambda \gg 1$ the left-hand side should be independent of Λ. However, individually the sum and the integral in the large round brackets diverge after integration over k. For this divergence to cancel the sum must be a close approximation to the integral. This occurs if we take $\mu^2 \simeq 1/4R^2$, because then the large round brackets may be written

$$\frac{1}{2\pi R^2} \sum_{l=0}^{\infty} \frac{l+\frac{1}{2}}{k^2 + R^{-2}(l+\frac{1}{2})^2} - \frac{1}{2\pi} \int_0^\infty \frac{k_\perp \, \mathrm{d}k_\perp}{k^2 + k_\perp^2} \tag{15}$$

which actually vanishes after integration over k. A more careful analysis shows that the solution of (13) is

$$\mu^2 = \frac{1}{4R^2} - \frac{4}{\pi \lambda_0 R^3} + O(R^{-4}). \tag{16}$$

For arbitrary d between 2 and 4, the same cancellation happens, with $l(l+1)$ replaced by $l(l+d-2)$, and so

$$\mu^2 \sim (d-2)^2/4R^2 \qquad |2 < d < 4|. \tag{17}$$

This verifies our general result $\xi^{-1} = x/R$ since $x = \frac{1}{2}(d-2+\eta)$, and for the $n \to \infty$ limit $\eta = 0$.

In conclusion we have shown how the universal amplitude relation for finite width two-dimensional strips generalises to higher dimensions. Whether this will provide a useful numerical approach to critical exponents remains to be seen.

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